

Dual Power Assignment via Second Hamiltonian Cycle

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Abstract

A *power assignment* is an assignment of transmission power to each of the wireless nodes of a wireless network, so that the induced graph satisfies some desired properties. The *cost* of a power assignment is the sum of the assigned powers. In this paper, we consider the dual power assignment problem, in which each wireless node is assigned a high- or low-power level, so that the induced graph is strongly connected and the cost of the assignment is minimized. We improve the best known approximation ratio from $\frac{\pi^2}{6} - \frac{1}{36} + \epsilon \approx 1.617$ to $\frac{11}{7} \approx 1.571$.

Moreover, we show that the algorithm of Khuller et al. [11] for the strongly connected spanning subgraph problem, which achieves an approximation ratio of 1.61, is 1.522-approximation algorithm for symmetric directed graphs. The innovation of this paper is in achieving these results via utilizing interesting properties for the existence of a second Hamiltonian cycle.

1 Introduction

Given a set P of wireless nodes distributed in a two-dimensional plane, a *power assignment* (or a range assignment), in the context of wireless networks, is an assignment of transmission range r_u to each wireless node $u \in P$, so that the induced communication graph has some desired properties, such as strong connectivity. The *cost* of a power assignment is the sum of the assigned powers, i.e., $\sum_{u \in P} r_u^\alpha$, where α is a constant called the *distance-power gradient* whose typical value is between 2 and 5. A power assignment induces a (directed) *communication graph* $G = (P, E)$, where a directed edge (u, v) belongs to the edge set E if and only if $|uv| \leq r_u$, where $|uv|$ is the Euclidean distance between u and v . The communication graph G is *strongly connected* if, for any two nodes $u, v \in P$, there exists a directed path from u to v in G . In the standard power assignment problem, one has to find a power assignment of P such that (i) its cost is minimized, and (ii) the induced communication graph is strongly connected.

When the available transmission power levels for each wireless node are continuous in a range of reals, many researchers have proposed algorithms for the strong connectivity power assignment problem [5, 7, 8, 13, 14]. In particular, 2-approximation algorithms based on minimum spanning trees were proposed in [5, 13]. When the wireless nodes are deployed

in the 2-dimensional or the 3-dimensional space, the problem is known to be NP-hard [7, 13]. A survey covering many variations of the problem is given in [6].

In this paper, we study a dual power assignment version, in which each wireless node can transmit in one of two (*high* or *low*) transmission power levels. Let r_H and r_L denote the transmission ranges of the high- and low-transmission powers, respectively. Since assigning more wireless nodes with the high power level results in a larger power consumption, the objective in the dual power assignment problem is equivalent to minimizing the number of wireless nodes that are assigned high-transmission range r_H .

The dual power assignment (DPA) problem was shown to be NP-hard [3, 16]. Rong et al. [16] gave a 2-approximation algorithm, while Carmi and Katz in [3] gave a $9/5$ -approximation algorithm and a faster $11/6$ -approximation algorithm. Later, Chen et al. [4] proposed an $O(n^2)$ time algorithm with approximation ratio of $7/4$. Recently, Calinescu [2] improved this approximation ratio to ≈ 1.61 , using in a novel way the algorithm of Khuller et al. [11, 12] for computing a minimum strongly connected subgraph.

A related version asks for a power assignment that induces a connected (also called “symmetric” or “bidirected”) graph. This version is also known to be NP-hard. The best known approximation algorithm is based on techniques that were applied to Steiner trees, and achieves approximation ratio of $3/2$ [15].

1.1 Our results

We present a conjecture regarding an interesting characterization for the existence of a second Hamiltonian cycle and its applications. We prove the conjecture for some special cases that are utilized (i) to improve the best known approximation ratio for the DPA problem from $\frac{\pi^2}{6} - \frac{1}{36} + \epsilon \approx 1.617$ to $\frac{11}{7} \approx 1.571$, and (ii) to show that the algorithm of Khuller et al. [11] for the strongly connected spanning subgraph problem, which achieves a approximation ratio of 1.61, is 1.522-approximation algorithm for symmetric unweighted directed graphs. Moreover, the correctness of the aforementioned conjecture implies that the approximation algorithm of Khuller et al. is actually a $3/2$ -approximation algorithm in symmetric unweighted digraphs.

2 Second Hamiltonian Cycle

A cycle in a graph is Hamiltonian if it visits each node of the graph exactly once; if a graph contains such a cycle, it is called a Hamiltonian graph. Deciding whether a graph is Hamiltonian has been shown to be NP-hard. A Hamiltonian graph G contains a second Hamiltonian cycle (SECHAMCYCLE for short) if there exist two Hamiltonian cycles in G that are differed by at least one edge. A classic result of Smith [19] states that each edge in a 3-regular graph is contained in an even number of Hamiltonian cycles. Thomason [17] extended Smith’s theorem to all graphs in which all nodes have an odd degree (Thomason’s lollipop argument). In addition, Thomassen [18] showed that every Hamiltonian r -regular graph, where $r \geq 72$, contains SECHAMCYCLE. This bound on r was reduced to 23 by Haxell et al. [10].

All these related works have considered the existence of SECHAMCYCLE on the whole

set of nodes. In this section, we consider the existence of SECHAMCYCLE also with respect to a subset of the nodes.

Let $G = (V, E)$ be a connected graph and let Γ be a subset of V . We say that G contains a Hamiltonian cycle on Γ if there exists a simple cycle in G whose nodes are exactly the nodes of Γ , i.e., the subgraph induced by Γ is a Hamiltonian graph. A cycle in G is Γ -Hamiltonian with respect to Γ if there exists a subset of nodes $U \subseteq (V \setminus \Gamma)$ such that G contains a Hamiltonian cycle on $\Gamma \cup U$. We denote such a cycle by $H_G(\Gamma)$; If G contains $H_G(\Gamma)$, then it is called a Γ -Hamiltonian graph. Moreover, we say that G contains a second Γ -Hamiltonian cycle (SEC- Γ -HAMCYCLE for short), if G contains a Hamiltonian cycle H on Γ and a Γ -Hamiltonian cycle $H_G(\Gamma)$, that are differed by at least one edge.

Fleischner [9] constructed a 3-regular graph G that has a dominating cycle Γ , such that no other SEC- Γ -HAMCYCLE exists. Below, we conjecture that replacing the regularity requirement with a connectivity requirement, implies the existence of SEC- Γ -HAMCYCLE.

Conjecture 2.1. *Let $G = (V, E)$ be a connected graph and let $\Gamma \subseteq V$, such that G contains a Hamiltonian cycle H on Γ and the graph $(V, E \setminus H)$ is connected. Then G contains a SEC- Γ -HAMCYCLE.*

The following conjecture, which is a special case of Conjecture 2.1, is shown in Lemma 2.14 to be actually equivalent, i.e., the correctness of Conjecture 2.2 yields the correctness of Conjecture 2.1.

Conjecture 2.2. *Let H be a Hamiltonian cycle on a set of nodes V . Every connected bipartite graph $G_b = (V, U, E)$ admits that the graph $G = (V \cup U, H \cup E)$ contains a SEC- V -HAMCYCLE.*

Notice that if two consecutive nodes in H share a common adjacent node of U in G_b , then Conjecture 2.2 is obviously true. Thus, we assume that no such two nodes exist. In addition, since nodes of U of degree 1 (in G_b) can be removed without affecting the correctness of the conjecture, we may assume that each node in U is of degree at least 2. Finally, we may assume that G_b is a tree. In the following lemmas, we prove Conjecture 2.2 for some special cases that are essential for proving Theorem 3.14 in the sequel section.

Lemma 2.3. *If each node in U is of degree 2, then the conjecture is true.*

Proof. Since each $u \in U$ is connected to two nodes of V , G_b can be converted to a spanning tree $T = (V, E_T)$ of V by connecting any two adjacent nodes of a node $u \in U$ via an edge and deleting u and the edges incident to it; that is, $G = (V, H \cup E_T)$. We distinguish two cases:

- **$|V|$ is even:** decompose T into a forest $T' = (V, E_{T'})$ s.t. each node of V has an odd degree in T' . The existence of such a decomposition can be easily proven by induction on $|V|$. The graph $G' = (V, H \cup E_{T'})$ is a Hamiltonian graph with nodes of odd degree; therefore, by Thomason's lollipop argument [17], it contains a SECHAMCYCLE on V that yields a SECHAMCYCLE on V in G .

- **$|V|$ is odd:** duplicate G to get a new graph $G_d = (V \cup V', H \cup E' \cup E_T \cup E'_T)$, in which V' is a copy of V , and, for each edge $\{v_i, v_j\} \in H$ (resp., $\{v_i, v_j\} \in E_T$), there is an edge $\{v'_i, v'_j\} \in E'$ (resp., $\{v'_i, v'_j\} \in E'_T$). Let v_i and v_j be two consecutive nodes in H . Connect v_i (resp., v_j) to its duplicated node v'_i (resp., v'_j) by an edge denoted by e_i (resp., e_j), and connect v_j to v'_i by an edge. Finally, remove from G_d the edges $\{v_i, v_j\}$ and $\{v'_i, v'_j\}$. The obtained graph G_d contains a Hamiltonian cycle and a spanning tree on $V \cup V'$ that are edge disjoint. By case 1, since $|V \cup V'|$ is even, we conclude that G_d contains a SECHAMCYCLE on $V \cup V'$; that contains e_i and e_j , and yields a SECHAMCYCLE on V in G .

□

Claim 2.4. *Let $T = (V, U, E_T)$ be a bipartite spanning tree of $V \cup U$, s.t. $|V|$ is even and all nodes of U are of degree 2 or 3. Then, there exists a forest $T' = (V, U, E'_T)$, in which (i) $E'_T \subseteq E_T$, (ii) each node in V is of odd degree, and (iii) each node in U is of degree 2.*

Proof. The claim can be proven by an induction on the number of nodes of degree 3 in U . Consider a node $u \in U$ of degree 3 that is connected to three nodes v_i, v_j and v_k from V . Since $|V|$ is even, at least one of the three subtrees rooted at v_i, v_j and v_k (and not containing u) has an even number of nodes from V . Assume w.l.o.g. that the subtree rooted at v_i has an even number of nodes from V . Thus, removing the edge $\{u, v_i\}$ from T decomposes T into two subtrees each has less number of nodes of degree 3 from U than T . Once we have a forest of subtrees each has even number of nodes from V and each node from U has a degree 2, we can convert it to a forest T' as in case 1 in the proof of Lemma 2.3. □

By this claim and by Lemma 2.3, we have the following lemma.

Lemma 2.5. *If each node in U is of degree at most 3, then the conjecture is true.*

The following corollary obtained by applying the duplication technique from the proof of Lemma 2.3.

Corollary 2.6. *If each node in U is of degree at most 3, then, for any edge e of H , there exists a SEC-V-HAMCYCLE in G that contains e .*

Corollary 2.7. *Let H be a Hamiltonian cycle on a set of nodes V , and let $F = (V, U, E)$ be a bipartite forest, such that (i) each node in U is of degree at most 3, (ii) each tree in F contains an even number of nodes of V . Then, the graph $G = (U \cup V, H \cup E)$ contains a SEC-V-HAMCYCLE.*

Corollary 2.8. *Let H be a Hamiltonian cycle on a set of nodes V , and let $F = (V, U, E)$ be a bipartite forest, such that (i) each node in U is of degree at most 3, (ii) each tree in F contains an even number of nodes of V **except of exactly one tree**. Then, the graph $G = (U \cup V, H \cup E)$ contains a SEC-V-HAMCYCLE.*

Proof. Let $T_{\text{odd}} \in F$ be the tree that contains an odd number of nodes of V , and let (v_i, v_j) be an edge of H , such that $v_i \in T_{\text{odd}}$. Consider the duplication technique from the proof of Lemma 2.3. Instead of connecting v_i (resp., v_j) to its duplicated node v'_i (resp., v'_j), we connect v_i (resp., v_j) to v'_j (resp., v'_i) and v_i to v'_i . Then, by Corollary 2.7 we are done. □

Given a bipartite graph (V, U, E) , for a node $v \in V$ and a subset $W \subseteq U$, denote by $N_v(W)$ the set of neighbors of v in W , i.e., $N_v(W) = \{u \in W : \{u, v\} \in E\}$.

Lemma 2.9. *Let U' be the subset of U containing all nodes of degree at least 4. If there exist two consecutive nodes v_i, v_j of H such that $N_{v_i}(U') \cup N_{v_j}(U') = U'$, then the conjecture is true.*

Proof. Consider the graph G'_b that is obtained from G_b by the following modification. Recall that $N_{v_i}(U') \cap N_{v_j}(U') = \emptyset$. For each node $u' \in N_{v_i}(U')$ (resp., $u' \in N_{v_j}(U')$), and for each $v \in V \setminus \{v_i\}$ (resp., $v \in V \setminus \{v_j\}$) that is adjacent to u' , we add a new node u_v to U and update the set E to be $E \setminus \{\{u', v\}\} \cup \{\{v_i, u_v\}, \{u_v, v\}\}$ (resp., $E \setminus \{\{u', v\}\} \cup \{\{v_j, u_v\}, \{u_v, v\}\}$). Then, we remove the edges $\{v_i, u'\}$ (resp., $\{v_j, u'\}$) from E , and the node u' from U . The obtained graph G'_b is a connected bipartite graph and each node in U is of degree at most 3; therefore, by Corollary 2.6, the graph obtained by adding the edge set H to G'_b contains a SEC-V-HAMCYCLE that contains the edge $\{v_i, v_j\}$. Thus, G contains SEC-V-HAMCYCLE. \square

Corollary 2.10. *Let v_i and v_j be two nodes of H such that $N_{v_i}(U') \cup N_{v_j}(U') = U'$, If by removing the nodes on one of the two paths between v_i and v_j on H (and their incident edges) from G_b , the graph G_b remains connected, then the conjecture is true.*

Claim 2.11. *Let $v_i \in V$ be a node such that $|N_{v_i}(U)| = 1$ in $G = (V \cup U, E \cup H)$ (i.e., v_i is a leaf in the tree $(V \cup U, E)$), and let v_{i+1} and v_{i-1} be its two neighbors in H (i.e., $\{v_i, v_{i+1}\}, \{v_{i-1}, v_i\} \in H$). Let $G^* = (V^* \cup U^*, E^* \cup H^*)$ be a graph obtained from G by the following modifications. Assume $N_{v_i} = \{u\}$, see Figure 1.*

$$\begin{aligned} V^* &\leftarrow V \cup \{v_l, v_r\} \\ U^* &\leftarrow U \cup \{u'\} \cup \{u_j : \forall v_j \in (N_u(V) \setminus \{v_i\})\} \setminus \{u\} \\ E^* &\leftarrow E \cup \{\{v_l, u'\}, \{v_r, u'\}\} \\ &\quad \cup \{\{v_i, u_j\}, \{u_j, v_j\} : \forall v_j \in N_u(V)\} \\ &\quad \setminus \{\{u, v_j\} : \forall v_j \in N_u(V)\} \\ H^* &\leftarrow H \cup \{\{v_{i-1}, v_l\}, \{v_l, v_i\}, \{v_i, v_r\}, \{v_r, v_{i+1}\}\} \\ &\quad \setminus \{\{v_{i-1}, v_i\}, \{v_i, v_{i+1}\}\} \end{aligned}$$

Then, SEC-V-HAMCYCLE in G^* admits a SEC-V-HAMCYCLE in G .*

Proof. Let C^* be SEC-V*-HAMCYCLE in G^* , then if $\{v_l, v_i\}, \{v_i, v_r\} \in C^*$, then C^* admits a SEC-V-HAMCYCLE in G . Therefore, assume w.l.o.g., that $\{v_l, v_i\} \notin C^*$; thus, $\{v_l, u'\}, \{u', v_r\} \in C^*$. We distinguish between two cases:

- $\{v_i, v_r\} \in C^*$: The path $P^* = (v_{i-1}, v_l, v_r, v_i, u_j)$ is a path in C^* . Thus, by replacing the path P^* in C^* with the path (v_{i-1}, v_i, u) in G , we have SEC-V-HAMCYCLE in G .
- $\{v_i, v_r\} \notin C^*$: The cycle C^* contains two paths $P_1 = (v_{i-1}, v_l, v_r, v_{i+1})$ and $P_2 = (v_j, u_j, v_i, u'_j, v'_j)$. Thus, by replacing the paths P_1 and P_2 in C^* with the paths (v_{i-1}, v_i, v_{i+1}) and (v_j, u, v'_j) in G , respectively, we have SEC-V-HAMCYCLE in G .

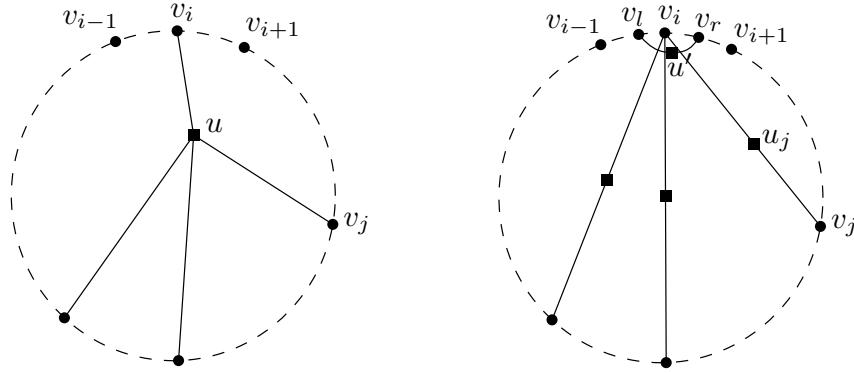


Figure 1: An illustration of the modified graph G^* (on the right) from graph G (on the left), where the edges of H and H^* are dashed, and the edges of E and E^* are solid.

□

In the next two lemmas we show that the conjecture holds for bounded values of $|V|$. First, we present a simple proof showing that the conjecture holds for $|V| \leq 15$, then, we provide a different proof that extends the bound to 23.

Lemma 2.12. *If $|V| \leq 15$, then the conjecture is true.*

Proof. Let U' be the set of nodes in U of degree at least 4. Recall that no two consecutive nodes v_i, v_{i+1} in H share a common adjacent node of U in G_b (i.e., $N_{v_i}(U) \cap N_{v_{i+1}}(U) = \emptyset$).

If there exists a node $v_i \in V$ such that $N_{v_i}(U') = U'$, then any adjacent node of v_i in H , w.l.o.g. v_{i+1} , satisfies $N_{v_i}(U') \cup N_{v_{i+1}}(U') = U'$, and, by Lemma 2.9, we are done. Thus, we may assume that no such a node exists, and hence, $|U'| > 1$.

Recall that we assume that $(V \cup U, E)$ is a tree, thus $|E| = |V| + |U| - 1 = 14 + |U|$. Moreover, $|E| \geq 4|U'| + 2|U \setminus U'| = 2|U'| + 2|U|$. Hence, $2|U'| + 2|U| \leq 14 + |U|$, and we have

$$2|U'| + |U| \leq 14. \quad (1)$$

This yields that $|U'| < 5$

We distinguish between the remaining 3 cases of U' cardinality.

- $|U'| = 2$: Since $|V| \leq 15$, by the pigeonhole principle, there are two consecutive nodes $v_i, v_{i+1} \in V$ such that $N_{v_i}(U') \cup N_{v_{i+1}}(U') = U'$, and, by Lemma 2.9, we are done.
- $|U'| = 3$: By (1), we have $|U \setminus U'| \leq 5$. Moreover, the tree $(V \cup U, E)$ has at least 8 leaves. Thus, by the pigeonhole principle, there exists a node $v \in V$ such that $|N_v(U)| = |N_v(U')| = 1$ (i.e., v is a leaf in the tree $(V \cup U, E)$), and two consecutive nodes $v_i, v_{i+1} \in V \setminus \{v\}$, such that $N_{v_i}(U') \cup N_{v_{i+1}}(U') = U' \setminus N_v(U')$. Then, by Claim 2.11 and by Lemma 2.9, we are done.

- $|U'| = 4$: By (1), we have $|U \setminus U'| \leq 2$. Moreover, the tree $(V \cup U, E)$ has at least 10 leaves. Thus, by the pigeonhole principle, there exist two nodes $v, v' \in V$ such that $|N_v(U)| = |N_v(U')| = 1$, $|N_{v'}(U)| = |N_{v'}(U')| = 1$ and $N_v(U') \neq N_{v'}(U')$, and two consecutive nodes $v_i, v_{i+1} \in V \setminus \{v, v'\}$, such that $N_{v_i}(U') \cup N_{v_{i+1}}(U') = U' \setminus (N_v(U') \cup N_{v'}(U'))$. Then, by Claim 2.11 and by Lemma 2.9, we are done.

□

In the following lemma we prove that the conjecture holds for $|V| < 24$. Actually, we show a stronger claim, that is, we claim that the conjecture holds also for wider family of graphs denoted \mathcal{G} . Let \mathcal{G} be the family of all graphs $(V \cup U, H \cup E)$, such that (V, U, E) is a bipartite graph, where $(V \cup U, E)$ is a forest and

- (i) each tree in $(V \cup U, E)$ has an even number of nodes of V ,
- (ii) H is a Hamiltonian cycle on the set of nodes V , and
- (iii) $|V| < 24$.

Notice that, if each graph in \mathcal{G} contains a second Hamiltonian cycle, then this implies that the conjecture is true for the original family of graphs (where $(V \cup U, E)$ is a tree) having $|V| < 24$.

Lemma 2.13. *The conjecture holds for each $G \in \mathcal{G}$.*

Proof. We prove the lemma by considering a minimal graph in \mathcal{G} that violates the conditions in the above lemmas, claims, and corollaries. More precisely, assume that there is a graph in \mathcal{G} that does not contain a second Hamiltonian cycle, and let $G = (V \cup U, H \cup E)$ be a graph in \mathcal{G} that does not contain a second Hamiltonian cycle, such that the number of nodes in U of degree at least 3 is minimal. Let $U' \subseteq U$ be the set of nodes of degree at least 4. Recall that each node of U is of degree at least 2. By the proof of Claim 2.4, the set U does contain a node of an odd degree, where the proof shows how to reduce the number of nodes of an odd degree (if exists), which contradicts the minimality of the number of nodes in U of degree at least 3.

By Lemma 2.5, if $|U'| = 0$, then G contains a second Hamiltonian cycle, in contradiction, and, by Lemma 2.9, there are no two consecutive nodes v_i, v_{i+1} in H such that $N_{v_i}(U') \cup N_{v_{i+1}}(U') = U'$. Therefore, $U' = \{u_1, \dots, u_k\}$, where $k \geq 2$. Moreover, by Claim 2.11, for each $v \in N_{u_i}(V)$, we have $|N_v(U)| > 1$ (i.e., v is not a leaf in $(V \cup U, E)$), where $u_i \in U'$. Furthermore, if $|U'| = 2$ (i.e., $U' = \{u_1, u_2\}$) and u_1 and u_2 do not belong to the same tree in $(V \cup U, E)$, then, clearly, $|V| \geq 24$, see Figure 2 for illustration. Otherwise, let $v \in N_{u_1}(V)$ and $v' \in N_{u_2}(V)$ be two nodes, such that v and v' are consecutive nodes in H , or one of the two paths between v and v' in H consists only of nodes that are leaves in $(V \cup U, E)$. Notice that there are at least two such pairs v and v' . By Corollary 2.10, G contains a second Hamiltonian cycle, in contradiction. Thus, V must contain at least one additional node for such a pair. Therefore, we have that $|V| \geq 24$.

Notice that, by extending the aforementioned to the case where $|U'| \geq 3$, we get that $|V|$ is at least 30 (i.e., the minimal graph that follows the above (where $|U'| \geq 3$) has a

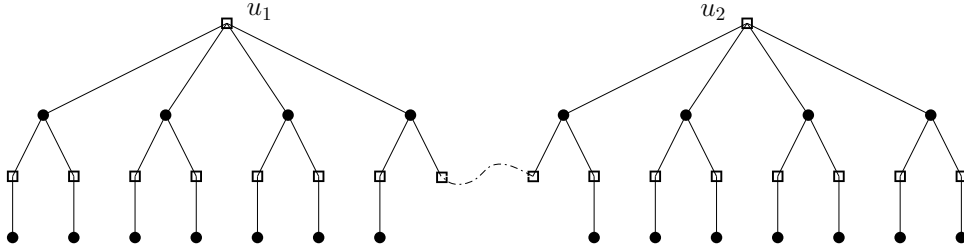


Figure 2: A minimal graph $(V \cup U, E)$ (with respect to $|U'|$) that does not admit a second Hamiltonian cycle by Lemma 2.5, Lemma 2.9, and Claim 2.11. The circles denote the nodes of V and the squares denote the nodes of U . The set U contains at least two nodes (u_1, u_2) of degree at least 4, each connected to non-leaf nodes of V .

tree of at least 29 nodes of V , however since it needs to be of even number of nodes of V , we conclude that $|V| \geq 30$). Thus, we assume that $|U'| = 2$ (i.e., $U' = \{u_1, u_2\}$). \square

In order to apply these lemmas for proving Theorem 3.14 it is sufficient to prove the following auxiliary lemma.

Lemma 2.14. *Let $G_{sb} = (V \cup U, E_{sb})$ be a connected graph such that V is an independent set in G_{sb} , and let $H = (V, E)$ be a Hamiltonian cycle on V . Then, the graph G_{sb} can be converted to a connected bipartite graph $G_b = (V, U^*, E_b)$ such that $U^* \subseteq U$ and, if the graph $G^* = (V \cup U^*, E \cup E_b)$ contains a SEC- V -HAMCYCLE, then $G = (V \cup U, E \cup E_{sb})$ also contains a SEC- V -HAMCYCLE.*

Proof. Let $G_U = (U, E_U)$ be the subgraph of G_{sb} that is induced by U , and let n be the number of edges in E_U . The proof is by induction on n .

Basis: $n = 0$, the claim clearly holds ($G_b = G_{sb}$).

Inductive step: Let $\{u_i, u_j\} \in E_U$, such that u_i is connected to at least one node $v \in V$. There exists such a node u_i , since the graph G_{sb} is connected. Consider the graph $G_{sb}^* = (V, U^*, E_{sb}^*)$ that is obtained from G_{sb} by connecting the adjacent nodes of u_i to u_j , and removing u_i and the edges incident to it, that is,

$$\begin{aligned} U^* &= U \setminus \{u_i\} \quad \text{and} \\ E_{sb}^* &= E_{sb} \cup \{\{u_j, w\} : \forall w \in N_{u_i}(U \cup V)\} \\ &\quad \setminus \{\{u_i, w\} : \forall w \in N_{u_i}(U \cup V)\}. \end{aligned}$$

By the induction hypothesis, G_{sb}^* can be converted to a connected bipartite graph $G_b = (V, U^*, E_b)$ satisfying the lemma. Thus, since any SEC- V -HAMCYCLE C^* in the graph $(V \cup U^*, E \cup E_{sb}^*)$ contains at most two edges that are incident to u_j and were generated during the modification of G_{sb} , the cycle C^* admits a SEC- V -HAMCYCLE in $G = (V \cup U, E \cup E_{sb})$. \square

3 Dual Power Assignment

Let P be a set of wireless nodes in the plane and let $G_R = (P, E_R)$ be the communication graph that is induced by assigning a high transmission range r_H to the nodes in a given subset $R \subseteq P$ and assigning low transmission range r_L to the nodes in $P \setminus R$, and with edge set $E_R = \{(u, v) : |uv| \leq r_u\}$.

Definition 3.1. A **strongly connected component** C of G_R is a maximal subset of P , such that for each pair of wireless nodes u, v in C , there exists a path from u to v in G_R .

Definition 3.2. The **components graph** CG_R of G_R is an undirected graph in which there is a node C_i for each strongly connected component C_i of G_R (throughout this paper, for convenience of presentation, we will refer to the nodes of CG_R as components, and to the wireless nodes of G_R as nodes). In addition, there exists an edge between two components C_i and C_j if and only if there exist two nodes $u \in C_i$ and $v \in C_j$ such that $|uv| \leq r_H$.

Definition 3.3. A set $Q \subseteq P$ is a **k -contracted set** of a set $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ of k distinct components in CG_R if $|Q \cap C_i| = 1$ for each $C_i \in \mathcal{C}$, and the components in \mathcal{C} are contained in the same strongly connected component in $G_{R \cup Q}$; see Figure 3 for illustration.

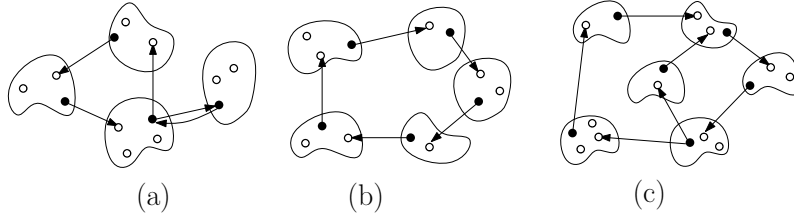


Figure 3: Examples of k -contractible structures: (a) 4-contractible structure, (b) 5-contractible structure, and (c) 6-contractible structure. The solid circles in each k -contractible structure represent the nodes of the k -contracted set of the components.

Let $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ be a set of components in CG_R , let Q be a k -contracted set of \mathcal{C} , and let v_i be the node in $Q \cap C_i$, for each $C_i \in \mathcal{C}$.

Definition 3.4. A **k -contractible structure** induced by \mathcal{C} and Q is a graph over \mathcal{C} in which there exists a directed edge from C_i to C_j if v_i can reach a node in C_j ; see Figure 3 for illustration.

Definition 3.5. A **leaf** in a k -contractible structure induced by \mathcal{C} and Q is a component $C_i \in \mathcal{C}$ such that (i) $\mathcal{C} \setminus \{C_i\}$ and $Q \setminus \{v_i\}$ induce a $(k-1)$ -contractible structure, (ii) each component in \mathcal{C} is reachable from C_i only via a path containing components from \mathcal{C} , and (iii) for each node u in C_i , by assigning a high transmission range to u , if u reaches a component from \mathcal{C} then every component $C \notin \mathcal{C}$ is not reachable from u .

Given a set P of n wireless nodes in the plane and two transmission ranges r_L and r_H such that the communication graph G_P that is induced by assigning a high transmission

range r_H to the nodes in P is strongly connected, in the *dual power assignment* problem the objective is to find a minimum set $R^* \subseteq P$ such that the induced communication graph G_{R^*} is strongly connected. Let OPT denote the size of R^* . We present an approximation algorithm that computes a set $R \subseteq P$, such that the graph G_R is strongly connected and the size of R is at most $\frac{11}{7} \cdot OPT$.

3.1 Approximation algorithm

Our algorithm is composed of an initialization and three phases and is based on the idea of Carmi and Katz [3] and Calinescu [2]. The main innovation of this algorithm is in achieving a better approximation ratio by utilizing the existence of a second Hamiltonian cycle. During the execution of the algorithm, we incrementally add nodes to the set R and update the graph G_R accordingly. The algorithm works as follows.

Initialization. Set $R = \emptyset$ and compute the induced communication graph G_R , i.e., $G_\emptyset = (P, E)$, by assigning r_L to each node in P and setting $E = \{(v, u) : |vu| \leq r_L\}$.

Phase 1. While G_R contains a j -contracted set, for $j \geq k$ (where k is a constant to be specified later), find a j -contracted set, add its j nodes to R , and update G_R accordingly.

Phase 2. Intuitively, we look for contractible structures, where we give priority to those with leaves and then according to their size. More precisely, for each iteration $i = k-1, k-2, \dots, 5, 4$, while G_R contains an i -contracted set, find a contracted set in the following priority order (where 1 is the highest priority), add its nodes to R , and update G_R accordingly (notice that, in each iteration i , any contractible structure in G_R is of size at most i).

1. A j -contracted set that induces a contractible structure with at least two leaves, where $j \geq 4$.
2. A j -contracted set that induces a contractible structure with one leaf, such that, if $i > \lceil k/2 \rceil$ then $j \geq \lceil k/2 \rceil$, otherwise $j = i$.
3. An i -contracted set that induces a contractible structure forming a simple cycle.
4. An i -contracted set that induces a contractible structure of combined cycles.

Phase 3. Find a minimum set $R_3^* \subseteq P$ such that $G_{R \cup R_3^*}$ is strongly connected, and update R to be $R \cup R_3^*$. Notice that at the beginning of this phase, any contracted set in G_R is of size at most 3. In Section 3.4 we show how to find an optimal solution R_3^* for such graphs in polynomial time.

The output of the algorithm is the set R , where the resulting graph G_R is strongly connected. In the following section, we analyze the performance guarantee of our algorithm.

3.2 Time complexity

An i -contracted set can be found naively in $O(n^{i+2})$ time by considering all combinations of sets of nodes of size i . Moreover, given a constant k finding a contracted set of size greater than k can be found in $O(n^{k+2})$ time. For example, a contracted set of size greater

than k that induces a simple cycle can be found by considering all paths of length k then by checking whether there is a simple path between the path's end-points that avoids the inner nodes of the path. Finally, since each contracted set reduces the number of components by at least two, the number of contracted sets found by algorithm is $O(n)$. Thus, the running time of the algorithm is polynomial. Notice that for a constant k , a k -contracted set can be found efficiently using ideas from Alon et al. [1], where they show how to find simple paths and cycles of a specified length k , using the method of *color-coding*.

3.3 Approximation ratio

In this section, we prove that the size of R (denoted by $|R|$) at the end of the algorithm is at most $\frac{11}{7} \cdot OPT$. Let R_i denote the set R at the beginning of the $k - i$ iteration of phase 2, for $4 \leq i \leq k - 1$, and let R_3 denote the set R at the beginning of phase 3. Given a set R_i , let n_i denote the number of components of CG_{R_i} , and let $OPT(G_{R_i})$ denote the size of a minimum set of nodes $R_i^* \subseteq P$ for which $G_{R_i \cup R_i^*}$ is strongly connected (i.e., R_i^* is an optimal solution for G_{R_i}). Let b_i (resp., $b_{i,j}$) denote the number of i -contracted sets (resp., j -contracted sets) found by the algorithm in the $k - i$ iteration. The following lemma immediately holds by Definition 3.3.

Lemma 3.6. *For each $4 \leq i < k$, we have*

$$n_i = n_{i-1} + (i-1) \cdot b_i + \sum_{j=4}^{i-1} (j-1) \cdot b_{i,j}.$$

Lemma 3.7. *For each $3 \leq i < k$, we have*

$$\frac{i}{i-1}(n_i - 1) \leq OPT(G_{R_i}) \leq 2(n_i - 1).$$

Proof. Let T be a spanning tree of CG_{R_i} . For each $\{C_i, C_j\} \in T$, select two nodes $v_i \in C_i$ and $v_j \in C_j$ such that $|v_i v_j| \leq r_H$, and add them to R_i . Clearly, the resulting communication graph is strongly connected and the cost of this solution is at most $2(n_i - 1)$, which proves the upper bound. The amortized cost of each contracted component of an i -contracted set is $\frac{i}{i-1}$. Hence, the lower bound follows. (The proof of this lemma also appears in previous related papers such as [3, 4, 16].) \square

Intuitively, the main ingredient of the algorithm is the way we select our contracted sets, which guarantees that each contracted set that is found in G_R saves high transmission range assignments for an optimal solution for G_R . Below we formalize this ingredient.

Let $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ be a set of k components in CG_R , let Q be a k -contracted set of \mathcal{C} , let v_j be the node in $Q \cap C_j$ for each $C_j \in \mathcal{C}$, and let S be a k -contractible structure induced by Q .

Observation 3.8. *Let ℓ be the number of leaves in S . Then,*

$$OPT(G_{R \cup Q}) \leq OPT(G_R) - \ell.$$

Corollary 3.9. Let \mathcal{L}_i denote the number of leaves contracted in the $k-i$ iteration. Then,

$$OPT(G_{R_{i-1}}) \leq OPT(G_{R_i}) - \mathcal{L}_i.$$

Observation 3.10. In $G_{R \cup Q}$, if there exists a node v in an optimal solution for G_R that induces only edges of the clique over \mathcal{C} (i.e., v reaches only components of \mathcal{C}), then $OPT(G_{R \cup Q}) < OPT(G_R)$.

Observation 3.11. Let $v_i \in C_i \cap Q$ be a node that reaches only one component $C_j \in \mathcal{C}$; then (i) any path from C_i to C_j via $C' \notin \mathcal{C}$ in CG_R must contain $C_k \in \mathcal{C}$, and (ii) for each node $u \in C_i$, by assigning a high transmission range to u , if u reaches C_j then every component $C' \notin \mathcal{C}$ is not reachable from u .

For simplicity of presentation we prove Lemma 3.12 and Lemma 3.13 for $k = 8$, therefore, the approximation ratio we obtained is based on $k = 8$. However, even-though we prove the lemmas for $k = 8$, the lemmas hold for greater values of k , therefore, we keep the statements of the lemmas in a general formulation.

Let Q_i denote an i -contracted set that is found during the $k-i$ iteration, and let S_i denote the contractible structure induced by Q_i . Recall that b_i (resp., $b_{i,j}$) denote the number of i -contracted sets (resp., j -contracted sets) found by the algorithm in the $k-i$ iteration.

Lemma 3.12. For each $4 \leq i \leq \lceil k/2 \rceil$, we have

$$OPT(G_{R_{i-1}}) \leq OPT(G_{R_i}) - 2b_i - 2 \sum_{j=4}^{i-1} b_{i,j}.$$

Proof. Recall that we put $k = 8$. Thus, $i = 4$ and $\sum_{j=4}^{i-1} b_{i,j} = 0$. Let $G_{R'_4}$ be the graph in which a contractible structure S_4 is found. We need to show that S_4 saves two to OPT of the remain graph, that is $OPT(G_{R'_4}) \geq OPT(G_{R'_4 \cup Q_4}) + 2$. If S_4 has two leaves, then, by Observation 3.8, S_4 saves two to $OPT G_{R'_4}$. Therefore, S_4 has at most one leaf and there are two such contractible structures, and, since there is no contractible structures of size greater than 4 in G_{R_i} (and in particular in $G_{R'_4}$), S_i saves two to $OPT(G_{R'_4})$. \square

Lemma 3.13. For each $\lceil k/2 \rceil < i < k$, we have

$$OPT(G_{R_{i-1}}) \leq OPT(G_{R_i}) - b_i - 2 \sum_{j=4}^{\lceil k/2 \rceil} b_{i,j} - \sum_{j=\lceil k/2 \rceil+1}^{i-1} b_{i,j}.$$

Proof. By Observation 3.8, we are left with providing a proof for contractible structures S_i without leaves, where $\lceil k/2 \rceil < i < k$. Let $G_{R'_i}$ be the graph in which S_i is found. First, we consider the case where S_i is a simple cycle ($5 \leq i \leq 7$), and assume towards a contradiction that $OPT(G_{R'_i}) = OPT(G_{R'_i \cup Q_i})$.

Let $H = (\mathcal{C}, E_H)$ be the undirected version of S_i in $CG_{R'_i}$, and let R_i^* be an optimal solution for $G_{R'_i}$. Let G be a spanning subgraph of $CG_{R'_i}$, where there is an edge in G

between $C_l \in CG_{R'_i}$ and $C_j \in CG_{R'_i}$ if there exists a node $v_l \in R'_i \cap C_l$ that can reach a node in C_j via high transmission range.

If there exist $C_l \in \mathcal{C}$ and $v \in R'_i \cap C_l$, such that v can reach only components in \mathcal{C} , then by Observation 3.10 we are done. Otherwise, let $G' = (\mathcal{C} \cup U', E')$ be a minimum subgraph of G in which all the components in \mathcal{C} are connected, where U' and E' are sets of components and edges in $CG_{R'_i}$, respectively. Let U be an empty set of nodes. For each edge $\{C_l, C_j\}$ of G' , such that $C_l, C_j \in \mathcal{C}$, we add a new node $u_{l,j}$ to U and update the set E' to be $E' \cup \{\{C_l, u_{l,j}\}, \{u_{l,j}, C_j\}\} \setminus \{C_l, C_j\}$. The obtained graph $G' = (\mathcal{C} \cup U' \cup U, E')$ is a connected graph where \mathcal{C} is an independent set. By Lemma 2.12 and Lemma 2.14, the graph $(\mathcal{C} \cup U \cup U', E' \cup E_H)$ contains a SEC- \mathcal{C} -HAMCYCLE H' . If H' contains nodes from U' , then H' admits a contracted structure of size at least $i + 1$ in $G_{R'_i}$, in contradiction. Otherwise, H' contains only the nodes of \mathcal{C} and nodes from U . Let $H_{\mathcal{C}}$ be the cycle obtained from H' by replacing each pair of consecutive edges $\{C_l, u_{l,j}\}, \{u_{l,j}, C_j\}$, where $u_{l,j} \in U$ and $C_l, C_j \in \mathcal{C}$, by the edge $\{C_l, C_j\}$. Recall that $\{C_l, C_j\}$ is an edge in $G_{R'_i}$ and for each $C_l \in \mathcal{C}$, there exists a node $v \in R'_i \cap C_l$, such that v can reach a component $C' \notin \mathcal{C}$ (i.e., $C' \in U'$). Thus, the cycle $H_{\mathcal{C}}$ with the component C' is a contracted structure of size at least $i + 1$ in $G_{R'_i}$, in contradiction.

We now consider contractible structure S_i ($i < 8$) that is neither a simple cycle nor a structure with leaves, that is S_i is a contractible structure of combined (overlapping) simple cycles $\{C_1, C_2, \dots, C_m\}$. W.l.o.g., let $C_1 \subset S_i$ be a simple cycle such that $\{C_2, \dots, C_m\}$ is a contractible structure. Let $\mathcal{C} = C_1$ and $\mathcal{C}' = \{C_2, \dots, C_m\}$. Moreover, let (C_1, \dots, C_t) be the components in $\mathcal{C} \setminus \mathcal{C}'$, such that there exists a component in \mathcal{C}' that has a directed edge to C_1 and there is a directed edge from C_t to a component in \mathcal{C}' , see Figure 4 for illustration.

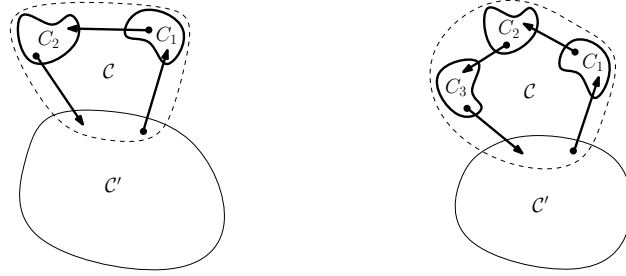


Figure 4: Illustration of two contractible structures, where on the left a contractible structure with a leaf, and on the right a contractible structure of combined cycles.

Notice that, if $1 \leq t \leq 2$, then C_t is a leaf, thus S_i is a contractible structure with a leaf. However, contractible structures with a leaf have already been considered, therefore $t \geq 3$. Moreover, since $i < 8$ and $t \geq 3$, \mathcal{C}' is also a simple cycle, thus the number of components in $(\mathcal{C}' \setminus \mathcal{C})$ is at least 3 (otherwise, S_i is a contractible structure with a leaf). Therefore, the number of components in $(\mathcal{C} \setminus \mathcal{C}')$ and $(\mathcal{C}' \setminus \mathcal{C})$ is exactly 3 (i.e. $t = 3$).

For $C_i \in \mathcal{C}$, let $\delta(C_i, \mathcal{C}')$ be the path that connects C_i to \mathcal{C}' in the optimal solution. Then, $\delta(C_3, \mathcal{C}')$ is the path that connects C_3 to \mathcal{C}' in the optimal solution. Consider the

three cases of $\delta(C_3, \mathcal{C}')$.

- $\delta(C_3, \mathcal{C}')$ does not pass through C_1 nor through C_2 , then it must go directly to a component in \mathcal{C}' (otherwise, we have a contractible structure of size greater than i), thus, by Observation 3.10, it saves one to $OPT(G_{R_i})$.
- $\delta(C_3, \mathcal{C}')$ passes through C_2 . C_3 can not go through another component $C_x \notin S_i$ to C_2 since in this case, by replacing the edge (C_2, C_3) with the reverse path from (C_3, C_2) that goes through C_x , we obtain a contractible structure of size greater than i . Moreover, C_3 can not go to C_2 and to another component $C_x \notin S_i$, since the reverse order of \mathcal{C} with C_x admits a contractible structure with a leaf of size at least 5. However, contractible structures with a leaf have already been considered. Thus, by Observation 3.10, we save one to $OPT(G_{R_i})$.
- $\delta(C_3, \mathcal{C}')$ passes through C_1 . Thus, there is a path δ_{C_3, C_1} from C_1 to C_3 that does not include components of $S_i \setminus \{C_1, C_3\}$. Denote by $\delta_{\overleftarrow{C_3, C_1}}$ the reverse path of δ_{C_3, C_1} . Consider $\delta(C_2, \mathcal{C}')$, following the same ideas of the two aforementioned cases, $\delta(C_2, \mathcal{C}')$ can not pass through neither C_1 , C_3 , nor directly to a component in \mathcal{C}' . Thus, $\delta(C_2, \mathcal{C}')$ goes through another component $C_x \notin S_i$. Therefore, by replacing the path (C_1, C_2, C_3, C) with the path $\delta_{\overleftarrow{C_3, C_1}}$, the edge (C_3, C_2) , and $\delta(C_2, \mathcal{C}')$, where $C \in \mathcal{C}'$ is a reachable component from C_3 in S_i , we obtain a contractible structure of size greater than i .

□

Theorem 3.14. *The aforementioned range assignment algorithm is an $11/7$ -approximation algorithm for the dual power assignment problem.*

Proof. Set k to be 8 and let n be the number of components of CG_\emptyset . By Lemma 3.7, $\frac{k}{k-1}$ is the amortized cost of each contracted component of a k -contracted set. Then, according to the algorithm description,

$$|R| \leq \frac{k}{k-1}(n - n_{k-1}) + \sum_{i=4}^{k-1} i \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} j \cdot b_{i,j} + OPT(G_{R_3}).$$

By Lemma 3.6, $n_{k-1} = n_3 + \sum_{i=4}^{k-1} (i-1) \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (j-1) \cdot b_{i,j}$, then

$$\begin{aligned} |R| &\leq \frac{k}{k-1} \left(n - n_3 - \sum_{i=4}^{k-1} (i-1) \cdot b_i - \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (j-1) \cdot b_{i,j} \right) \\ &\quad + \sum_{i=4}^{k-1} i \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} j \cdot b_{i,j} + OPT(G_{R_3}) \\ &= \frac{1}{k-1} \left(k \cdot n - k \cdot n_3 + \sum_{i=4}^{k-1} (k-i) \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (k-j) \cdot b_{i,j} + (k-1) \cdot OPT(G_{R_3}) \right), \end{aligned}$$

and, by Lemma 3.7, $OPT(G_{R_3}) \leq 2(n_3 - 1)$, then $n_3 \geq \frac{OPT(G_{R_3})}{2}$, and we have,

$$\begin{aligned} |R| &\leq \frac{1}{k-1} \left(k \cdot n + \sum_{i=4}^{k-1} (k-i) \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (k-j) \cdot b_{i,j} + (k-1 - \frac{k}{2}) \cdot OPT(G_{R_3}) \right) \\ &= \frac{1}{k-1} \cdot \left(k \cdot n + \sum_{i=4}^{k-1} (k-i) \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (k-j) \cdot b_{i,j} \right) + \frac{k-2}{2(k-1)} \cdot OPT(G_{R_3}), \end{aligned}$$

and, by Lemma 3.12 and Lemma 3.13,

$$\begin{aligned} OPT(G_{R_3}) &\leq OPT(G_{R_{k-1}}) - 2 \sum_{i=4}^{\lceil k/2 \rceil} b_i - 2 \sum_{i=5}^{\lceil k/2 \rceil} \sum_{j=4}^{i-1} b_{i,j} - \sum_{i=\lceil k/2 \rceil+1}^{k-1} b_i \\ &\quad - 2 \sum_{i=\lceil k/2 \rceil+1}^{k-1} \sum_{j=4}^{\lceil k/2 \rceil} b_{i,j} - \sum_{i=\lceil k/2 \rceil+1}^{k-1} \sum_{j=\lceil k/2 \rceil+1}^{i-1} b_{i,j}, \end{aligned}$$

then,

$$\begin{aligned} |R| &\leq \frac{1}{k-1} \cdot \left(k \cdot n + \sum_{i=4}^{k-1} (k-i) \cdot b_i + \sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (k-j) \cdot b_{i,j} \right) \\ &\quad + \frac{k-2}{2(k-1)} \cdot \left(OPT(G_{R_{k-1}}) - 2 \sum_{i=4}^{\lceil k/2 \rceil} b_i - 2 \sum_{i=5}^{\lceil k/2 \rceil} \sum_{j=4}^{i-1} b_{i,j} \right. \\ &\quad \left. - \sum_{i=\lceil k/2 \rceil+1}^{k-1} b_i - 2 \sum_{i=\lceil k/2 \rceil+1}^{k-1} \sum_{j=4}^{\lceil k/2 \rceil} b_{i,j} - \sum_{i=\lceil k/2 \rceil+1}^{k-1} \sum_{j=\lceil k/2 \rceil+1}^{i-1} b_{i,j} \right). \end{aligned}$$

Since

$$\frac{1}{k-1} \cdot \sum_{i=4}^{k-1} (k-i) \cdot b_i - \frac{k-2}{2(k-1)} \cdot \left(2 \sum_{i=4}^{\lceil k/2 \rceil} b_i + \sum_{i=\lceil k/2 \rceil+1}^{k-1} b_i \right) \leq 0$$

and

$$\begin{aligned} \frac{1}{k-1} \cdot \left(\sum_{i=5}^{k-1} \sum_{j=4}^{i-1} (k-j) \cdot b_{i,j} \right) &- \frac{k-2}{2(k-1)} \cdot \left(2 \sum_{i=5}^{\lceil k/2 \rceil} \sum_{j=4}^{i-1} b_{i,j} \right. \\ &\quad \left. + 2 \sum_{i=\lceil k/2 \rceil+1}^{k-1} \sum_{j=4}^{\lceil k/2 \rceil} b_{i,j} + \sum_{i=\lceil k/2 \rceil+1}^{k-1} \sum_{j=\lceil k/2 \rceil+1}^{i-1} b_{i,j} \right) \leq 0, \end{aligned}$$

we have

$$|R| \leq \frac{k}{k-1} \cdot n + \frac{k-2}{2(k-1)} \cdot OPT(G_{R_{k-1}}).$$

Finally, since $OPT(G_{R_0}) \geq n$ and $OPT(G_{R_0}) \geq OPT(G_{R_{k-1}})$, we have

$$|R| \leq \frac{3k-2}{2(k-1)} \cdot OPT(G_{R_0}),$$

thus, for $k = 8$, we have

$$|R| \leq \frac{11}{7} \cdot OPT.$$

□

3.4 Optimal solution for G_{R_3}

Given a set P of n wireless nodes in the plane, two transmission ranges r_L and r_H , and $R_3 \subseteq P$, such that G_{R_3} does not contain a contracted set of size greater than 3. Then finding a minimum set $R_3^* \subseteq P$, such that the induced communication graph $G_{R_3 \cup R_3^*}$ is strongly connected can be done in polynomial time.

Our algorithm is based on the idea of Carmi and Katz [3] and works as follows. Set $R = \emptyset$ and compute the induced communication graph G_{R_3} by assigning r_H to each node in R_3 and assigning r_L to each node in $P \setminus R_3$. Next, while $G_{R_3 \cup R}$ contains a 3-contracted set forming a simple cycle, find such a contracted set, and add its 3 nodes to R . When $G_{R_3 \cup R}$ does not contain a 3-contracted set forming a simple cycle, it induces a tree of well-separated j -contracted sets, we solve the subproblem in each strongly connected component of $G_{R_3 \cup R}$ independently, and add to R the nodes that are in the solution.

Notice that the resulting $CG_{R_3 \cup R}$ has one component, and, therefore, $G_{R_3 \cup R}$ is strongly connected. In the following, we prove that this algorithm solves the problem optimally, i.e., $|R| = |R_3^*|$.

Let $\mathcal{C} = \{C_1, C_2, C_3\}$ be a set of 3 components in $CG_{R_3 \cup R}$ and let Q be a 3-contracted set of \mathcal{C} , such that the 3-contractible structure induced by Q forms a simple cycle. The following two observations follow from the fact that the graph $CG_{R_3 \cup R}$ does not contain a contracted set of size greater than 3.

Observation 3.15. *By adding the nodes in Q to R the problem is separated into at least three independent subproblems. I.e., by removing the components in \mathcal{C} and the edges incident to them from $CG_{R_3 \cup R}$, the graph $CG_{R_3 \cup R}$ remains with at least three connected components.*

Observation 3.16. *There exists an optimal solution R_3^* for G_{R_3} that contains the nodes in Q .*

When $G_{R_3 \cup R}$ does not contain a 3-contracted set forming a simple cycle, it induces a tree of well-separated j -contracted sets. Thus, assigning a high transmission range to a node in one strongly connected component cannot result in forcing an assignment of a high transmission range to a node in another strongly connected component. Therefore, each strongly connected component of $G_{R_3 \cup R}$ is an independent subproblem. Each node in a strongly connected component of $G_{R_3 \cup R}$ can reach at most two other strongly connected components via high transmission range. Hence, each strongly connected component is an instance of the *2 set cover* problem, which can be solved optimally.

Thus, we conclude that the algorithm described above solves the problem optimally.

4 Application of a second Hamiltonian cycle to SCSS

In this section, we show that the correctness of Conjecture 2.2 implies that the approximation algorithm of Khuller et al. [11], which achieves a performance guarantee of ≈ 1.61 for the SCSS problem, is a $3/2$ -approximation algorithm in symmetric unweighted digraphs. This matches the best known approximation ratio for this problem, achieved by Vetta [20]. Even though Vetta's result is very novel, it is much more complicated.

Given a strongly connected graph, the algorithm finds a cycle of length at least some constant k while there exists such a cycle, and then a longest cycle in the current graph, contracts the cycle, and recurses. The contracted graph remains strongly connected. When the graph, finally, collapses into a digraph with cycles of length at most 3, it solves the subproblem optimally and returns the set of edges contracted during the course of the algorithm as the desired SCSS.

This algorithm differs from the DPA algorithm (described in Section 3) in the contracted structures. More precisely, only simple cycle structures are found (since simple cycle structures are the only contracted structures exist). Thus, assuming Conjecture 2.2 holds, each structure found during the algorithm saves at least two edges for an optimal solution. This implies the following lemma that is similar but stronger than Lemma 3.12.

Lemma 4.1. *For each $4 \leq i \leq k - 1$, we have $OPT(G_{i-1}) \leq OPT(G_i) - 2b_i$, where $OPT(G_i)$ is the size of an optimal solution for the component graph at the beginning of the $k - i$ iteration, and b_i is the number of contracted structures found and contracted by the algorithm in the $k - i$ iteration.*

By combining this lemma with Lemma 3.6 and Lemma 3.7, we get the following theorem.

Theorem 4.2. *The algorithm of Khuller et al. in [11] (described above) is a $3/2$ -approximation algorithm for the SCSS problem in symmetric unweighted digraphs, assuming Conjecture 2.2 holds.*

Proof. Applying a similar (yet simpler) analysis of the performance of the dual power assignment algorithm (Section 3) yields an upper bound of $\frac{3k-2}{2(k-1)}OPT$. This approximation ratio tends to $3/2$ as k increases. \square

Since we verified Conjecture 2.2 for $|V| < 24$ (see Lemma 2.13), we have the following corollary.

Corollary 4.3. *The algorithm of Khuller et al. in [11] is a $35/23$ -approximation algorithm (≈ 1.522) for the SCSS problem in symmetric unweighted digraphs.*

4.1 SCSS for symmetric digraphs with bounded cycle length

In [12], Khuller et al. consider the SCSS problem in a strongly connected digraphs with bounded cycle length. They give a proof that, for graphs where each directed cycle has at most three edges is equivalent to the maximum bipartite matching, and, thus can be solved optimally. Moreover, in [11] Khuller et al. prove that the problem remains NP-hard

even when the maximum cycle length is at most five. In this section, we consider the same problem in symmetric digraphs with bounded cycle length, and show the following.

Theorem 4.4. *The algorithm of Khuller et al. in [11] (described above) is a $\frac{3k-2}{2k}$ -approximation algorithm for the SCSS problem in symmetric unweighted digraphs, where k is the maximum cycle length in the graph, assuming Conjecture 2.2 holds or $k < 24$.*

Proof. The length of the longest cycle is at most k , thus the first phase of the algorithm (looking for cycles of length greater than k) is redundant. Therefore, we have

$$\begin{aligned}
|R| &\leq \sum_{i=4}^k i \cdot b_i + OPT(G_{R_3}) \\
&= \sum_{i=4}^k i \cdot b_i + \frac{OPT(G_{R_3})}{2} + \frac{OPT(G_{R_3})}{2} \\
&\stackrel{(1)}{\leq} \sum_{i=4}^k i \cdot b_i + \frac{OPT(G_R) - 2 \sum_{i=4}^k b_i}{2} + \frac{OPT(G_{R_3})}{2} \\
&\stackrel{(2)}{\leq} \sum_{i=4}^k i \cdot b_i + \frac{OPT(G_R) - 2 \sum_{i=4}^k b_i}{2} + \frac{2(n_3 - 1)}{2} \\
&\stackrel{(3)}{=} n - n_3 + \sum_{i=4}^k b_i + \frac{OPT(G_R)}{2} - \sum_{i=4}^k b_i + n_3 - 1 \\
&= n + \frac{OPT(G_R)}{2} - 1 \\
&\stackrel{(2)}{\leq} OPT(G_R) \cdot \left(\frac{k-1}{k} + \frac{1}{2} \right) \\
&= \frac{3k-2}{2k} \cdot OPT(G_R)
\end{aligned}$$

where (1) follows from Lemma 4.1, (2) follows from Lemma 3.7, and (3) follows from Lemma 3.6. □

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